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High frequency limiting virtual-mass coefficients of heaving half-immersed spheres

By A. M. J. DAVIS
Department of Mathematics, University College London

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High frequency surface waves are generated by the forced heaving of either two half-immersed spheres in infinite water or by a half-immersed sphere in a hemispherical lake. The virtual-mass coefficients can be found, to leading order, in terms of wave-free limit potentials.

1. Introduction

Two spheres S_1 and S_2 with comparable radii a and b are half-immersed in infinitely deep, incompressible, inviscid fluid under gravity. Cartesian co-ordinates (x, y, z) are chosen with z measured vertically downwards and the x axis along the line of centres of the spheres, i.e. the origin is in the undisturbed free surface F and for convenience is between the spheres. These are forced to heave with small constant amplitudes, possibly different, but the same period $2\pi/\sigma$ about the equilibrium position and the fluid motion generated is assumed small enough for the equations to be linearized. With surface tension also neglected, the velocity potential, which is of the form $\mathrm{Re}\,[\phi(x,y,z)\,e^{-i\sigma t}]$, satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \tag{1.1}$$

throughout the fluid and the boundary conditions

$$K\phi + \partial\phi/\partial z = 0 \quad \text{at} \quad z = 0, \tag{1.2}$$

where $K = \sigma^2/g$ and g is the gravitational acceleration, and

$$\partial(\phi - Uz)/\partial n = 0 \quad \text{on} \quad S_1, \tag{1.3}$$

$$\partial(\phi - \alpha Uz)/\partial n = 0 \quad \text{on} \quad S_2, \tag{1.3a}$$

where $\partial/\partial n$ denotes the normal derivative directed into the fluid and U and $|\alpha|\,U$ are the respective amplitudes of the heave velocities. On physical considerations, the complex factor α may be chosen as zero (S_2 fixed) or assigned a value such that $|\alpha|$ is neither large nor small compared with unity. The remaining condition on ϕ is that only outgoing waves be present at infinity.

Only short waves will be considered, in which case K^{-1} is small compared with the radius a of S_1 and the surface wave disturbance is essentially confined to a layer of thickness $O(K^{-1})$ below the free surface. The dimensionless quantity

$$N = Ka = \sigma^2 a/g \tag{1.4}$$

is therefore large and it is helpful to write

$$\phi = \phi_0 - \frac{1}{K}\frac{\partial \phi_0}{\partial z} + \frac{1}{N}\phi_1, \tag{1.5}$$

where ϕ_0 is the limit potential, satisfying (1.1), (1.3), (1.3a) and the limiting forms of (1.2) and the radiation condition as $K \to \infty$, namely

$$\phi_0 = 0 \quad \text{on} \quad F \quad \text{and at} \quad \infty. \tag{1.6}$$

Then ϕ_0 is wave free and, because $\phi_0 - K^{-1}\,\partial\phi_0/\partial z$ satisfies (1.2), the conditions on ϕ_1 are of the same type as those on ϕ. The justification for the introduction of ϕ_1 is that, according to (1.3) and (1.3a), $\partial\phi/\partial n$ vanishes at the intersections of the spheres with F. The methods of Davis (1976b) are, except for the coefficient of N^{-1} in (1.7), readily adapted for the presence of more than one such immersed body and hence the results are applicable to the current problem.

The virtual-mass coefficients V_1 and V_2 of the spheres S_1 and S_2 respectively are given by

$$V_1 \sim -\frac{3}{2\pi a^3 U^2}\,\mathrm{Re}\int_{S_1} \phi_0 \frac{\partial \phi_0}{\partial n}\,dS + O(N^{-1}), \tag{1.7}$$

$$V_2 \sim -\frac{3}{2\pi b^3 |\alpha|^2 U^2}\,\mathrm{Re}\int_{S_2} \phi_0 \frac{\partial \overline{\phi}_0}{\partial n}\,dS + O(N^{-1}) \quad (\alpha \neq 0).$$

The work done in one time cycle by each sphere on the fluid is

$$-\pi\rho\,\mathrm{Im}\int_{S_j} \phi \frac{\partial \overline{\phi}}{\partial n}\,dS \quad (j = 1, 2)$$

and has a non-zero first-order term, in general, since ϕ_0 is now complex. For this reason, the damping coefficients, defined like V_1 and V_2 but with the imaginary parts taken, are $O(1)$. But, by applying Green's theorem to ϕ and $\overline{\phi}$ throughout the fluid region, it follows that

$$-2\,\mathrm{Im}\sum_{j=1}^{2}\int_{S_j} \phi \frac{\partial \overline{\phi}}{\partial n}\,dS = -\frac{a}{N^3}\,\mathrm{Im}\lim_{R_0\to\infty}\int_{R=R_0}\left(\phi_1 \frac{\partial \overline{\phi}_1}{\partial R}\right)_F ds$$
$$= O(U^2 a^3 N^{-4}).$$

Hence, to the first few orders of magnitude, the input of energy from one sphere to the fluid is absorbed by the other sphere.

The general case outlined here is discussed in §5, the intermediate sections being concerned with some simplified situations. With $\alpha = 1$ and $b = a$, the spheres may touch or be separated, the latter case being better described as a sphere near a vertical wall. Also considered is a heaving sphere in a hemispherical lake, a situation where no energy can be lost. In the symmetric cases and for the single sphere, the suffixes are dropped from S and V.

It is useful to compare the present results with the rigorous results for a single sphere in infinite fluid (Davis 1971), viz.

$$V \sim \tfrac{1}{2} - 3/16N, \tag{1.8}$$

and for a sphere at the centre of a hemispherical lake (Davis 1975a), viz.

$$V \sim \tfrac{1}{2}[1 + 3/(\lambda^3 - 1)] + O(N^{-1}), \tag{1.9}$$

where λ (>1) is the ratio of the radii.

Owing to the more complicated geometries, only the terms of order 1 are sought here. However, unlike the corresponding two-dimensional problems involving cylinders (Davis 1976*a*), closed-form solutions are not available and the second-order differential and difference equations must be solved numerically, the results being displayed at various places in the subsequent text.

The limiting problems then considered are equivalent to those of a sphere in semi-infinite fluid, a sphere in a spherical container and two spheres in unbounded fluid, the motion being perpendicular to the line of centres. Early work on the last problem, using an approximation based on successive images, is described by Basset (1888, chap. 11) and, more briefly, by Lamb (1932, §§99, 138). Their formulae for kinetic energy correspond essentially to those for the virtual masses.

Motion along the line of centres is also included by these authors, and, being axisymmetric, provides a simpler problem to attack by the methods employed here. Such work has already been published by Majumdar (1961) for spheres in contact and by Weihs & Small (1975*a*) for separated spheres. Weihs & Small (1975*b*) also published work on the contact case but they represented the velocity potential by a summation rather than an integral and hence could not obtain a correct solution. Tangent-sphere and bispherical co-ordinates have also been used, for axisymmetric Stokes flow past two spheres, by Davis *et al.* (1976).

The potential flow past a sphere tangential to a plane was shown by Latta & Hess (1973), using the method of inversion, to have a velocity singularity of order $r^{\sqrt{2}-2}$ at the point of contact.

Thus the presentation adopted here, i.e. exploitation of co-ordinate systems to set up exact equations which are then solved numerically, appears to be new.

2. Equal spheres in contact

Here the spheres have a common radius a, touch at the origin and heave in phase with the same amplitude. Defining tangent-sphere co-ordinates (ξ, η, θ) by

$$x = \frac{2a\eta}{\xi^2+\eta^2}, \quad (y, z) = \frac{2a\xi}{\xi^2+\eta^2}(\sin\theta, \cos\theta), \tag{2.1}$$

the sphere boundaries are given by $\eta = \pm 1$, the free surface by $\theta = \pm\frac{1}{2}\pi$ (except on the x axis, where $\xi = 0$) and the fluid region by $0 \leqslant \xi \leqslant \infty$, $|\eta| \leqslant 1$, $|\theta| \leqslant \frac{1}{2}\pi$. The symmetry implies that $\phi(\xi, \eta, \theta)$ is an even function of η and conditions (1.3) and (1.6) on the limit potential ϕ_0 are now

$$\frac{\partial\phi_0}{\partial\eta} = -\frac{4aU\xi}{(\xi^2+1)^2}\cos\theta \quad \text{at} \quad \eta = 1, \tag{2.2}$$

$$\phi_0 = 0 \quad \text{when} \quad \xi\cos\theta = 0.$$

In these co-ordinates, Laplace's equation is reduced to cylindrical form by writing

$$\phi_0 = (\xi^2+\eta^2)^{\frac{1}{2}}\chi_0, \tag{2.3}$$

whence (1.1) becomes

$$\frac{\partial^2\chi_0}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\chi_0}{\partial\xi}+\frac{\partial^2\chi_0}{\partial\eta^2}+\frac{1}{\xi^2}\frac{\partial^2\chi_0}{\partial\theta^2} = 0.$$

Then χ_0 is evidently of the form

$$\chi_0 = Ua\cos\theta \int_0^\infty J_1(s\xi)\frac{\cosh s\eta}{s\sinh s}\alpha(s)\,ds, \tag{2.4}$$

where $\alpha(0) = 0$ since the integral is to be convergent. $\alpha(s)$ is determined from (2.2) and (2.3), which yield

$$\int_0^\infty J_1(s\xi)\frac{\cosh s}{s\sinh s}\alpha(s)\,ds + (\xi^2+1)\int_0^\infty J_1(s\xi)\,\alpha(s)\,ds = -\frac{4\xi}{(\xi^2+1)^{\frac{3}{2}}}. \tag{2.5}$$

Now, defining the operator $\mathscr{L}$ by

$$\mathscr{L} = \frac{d}{ds}\left[s\frac{d}{ds}\right] - \frac{1}{s},$$

it follows that

$$\xi^2\int_0^\infty J_1(s\xi)\,\alpha(s)\,ds = -\int_0^\infty \mathscr{L}[J_1(s\xi)]\frac{\alpha(s)}{s}\,ds$$
$$= -\int_0^\infty \mathscr{L}\left[\frac{\alpha(s)}{s}\right]J_1(s\xi)\,ds.$$

Since also

$$\int_0^\infty J_1(s\xi)\,s\,e^{-s}\,ds = \xi(\xi^2+1)^{-\frac{3}{2}},$$

the inversion of the Hankel transform in (2.5) shows that $\alpha(s)$ satisfies the differential equation

$$\frac{d^2\alpha}{ds^2} - \frac{1}{s}\frac{d\alpha}{ds} - \left(1 + \frac{1}{s}\coth s\right)\alpha = 4s\,e^{-s} \quad (s \geqslant 0). \tag{2.6}$$

At large s, $\alpha \sim \alpha_1$, with exponentially small error, where

$$\alpha_1'' - s^{-1}\alpha_1' - (1+s^{-1})\,\alpha_1 = 4s\,e^{-s}$$

and the exact solution for α_1 is

$$\alpha_1 = (A - s^2)\,e^{-s} + B(s - \tfrac{1}{2})\,e^{s}.$$

Since the exponentially growing solution is inadmissible, α must tend to zero as $s \to \infty$.

At small s, a particular solution of (2.6) is

$$\alpha = 2s^3e^{-s}(1 + \tfrac{5}{7}s + \tfrac{8}{21}s^2 + \ldots)$$

while the two series solutions in the complementary function have first terms $s^{1+\sqrt{2}}$ and $s^{-(\sqrt{2}-1)}$ respectively. The latter must be rejected here since $\alpha(0) = 0$.

It is readily shown that the vertical velocity on the z axis is given by

$$\left(\frac{\partial\phi_0}{\partial z}\right)_{x=y=0} = -\tfrac{1}{2}U\xi^2\int_0^\infty J_1(s\xi)\,\beta(s)\,ds \quad (\xi z = 2a),$$

where

$$\beta(s) = \frac{\alpha(s)}{s\sinh s} - \frac{d}{ds}\left[\frac{\alpha(s)}{\sinh s}\right].$$

The behaviour of the integral as $\xi \to \infty$ is determined by that of $\beta(s)$ at small s. Now $\alpha = O(s^{1+\sqrt{2}})$, i.e. $\beta = O(s^{-1+\sqrt{2}})$, as $s \to 0$; hence

$$\left(\frac{\partial\phi_0}{\partial z}\right)_{x=y=0} = O(\xi^{2-\sqrt{2}}) = O(z^{\sqrt{2}-2}) \quad \text{as} \quad z \to 0$$

and the velocity is found to be unbounded at the point of contact, in agreement with Latta & Hess (1973), whose solution by inversion is less simple than that presented here but, of course, involves a differential equation [their equation (13)] equivalent to (2.6).

Numerical computation was necessary to solve (2.6) for α and, in particular, to evaluate the integrals involving α which appear below.

The virtual-mass coefficient is given, from (1.7), (2.1) and (2.2), by

$$V \sim \frac{3}{2\pi a^3 U^2}\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (-\phi_0)_{\eta=1} \frac{4a\xi U\cos\theta}{(\xi^2+1)^2}\frac{2a\xi}{\xi^2+1}\,d\xi\,d\theta$$
$$= -6\int_0^\infty \frac{\xi^2}{(\xi^2+1)^{\frac{5}{2}}}\int_0^\infty J_1(s\xi)\frac{\coth s}{s}\alpha(s)\,ds\,d\xi$$

on substitution of (2.3) and (2.4). Using (2.5), this can be written as

$$V \sim 6\int_0^\infty \alpha(s)\,ds\int_0^\infty \frac{\xi^2 J_1(s\xi)}{(\xi^2+1)^{\frac{3}{2}}}\,d\xi + 24\int_0^\infty \frac{\xi^3}{(\xi^2+1)^4}\,d\xi$$
$$= 2\left[1+3\int_0^\infty \alpha(s)\,e^{-s}\,ds\right] = 0{\cdot}621. \tag{2.7}$$

Consider now the waves radiated to infinity. As $R = (x^2+y^2)^{\frac{1}{2}} \to \infty$,

$$\phi \sim N^{-1}\phi_1 \sim N^{-1}e^{-Kz}\psi(x,y)$$

(according to Davis 1976*b*), where ψ is the solution of the two-dimensional wave equation

$$(\nabla^2+K^2)\psi = 0 \quad \text{on} \quad F \quad (|\eta| \leqslant 1, 0 \leqslant \xi < \infty)$$

satisfying

$$\frac{\partial\psi}{\partial\eta} = 2a\frac{\partial}{\partial\eta}\left(\frac{\partial\phi_0}{\partial z}\right)_{z=0} \quad \text{at} \quad \eta = \pm 1$$

and the radiation condition

$$\lim_{R\to\infty} R^{\frac{1}{2}}(\partial\psi/\partial R - iK\psi) \to 0 \quad \text{as} \quad R\to\infty.$$

Leaving aside the interference between the spheres, the essential features of the distribution of wave radiation to infinity can be obtained from simple use of ray theory. Except in the shadow regions containing the x axis, there is propagated in each direction a ray from each sphere. Since the rays leave the spheres normally, the distribution of wave radiation is determined by the function $(\partial\psi/\partial n)_{\eta=\pm1}$. From (2.1), (2.3) and (2.4)

$$\left(\frac{\partial\phi_0}{\partial z}\right)_{z=0} = \frac{U}{2\xi}(\xi^2+\eta^2)^{\frac{3}{2}}\int_0^\infty J_1(s\xi)\frac{\cosh s\eta}{s\sinh s}\alpha(s)\,ds,$$

$$\frac{1}{U}\left(\frac{\partial\psi}{\partial n}\right)_{\eta=1} = -\frac{a}{U}(\xi^2+1)\left\langle\frac{\partial}{\partial\eta}\left(\frac{\partial\phi_0}{\partial z}\right)_{z=0}\right\rangle_{\eta=1}$$
$$= 6+\xi^{-1}(\xi^2+1)^{\frac{5}{2}}\int_0^\infty J_1(s\xi)\,\alpha(s)\,ds$$

after using (2.5). The values at $\xi = 0, 1$ are respectively

$$6+\int_0^\infty s\alpha(s)\,ds = 0{\cdot}109,$$

$$6+4\sqrt{2}\int_0^\infty J_1(s)\,\alpha(s)\,ds = 4{\cdot}574.$$

(The corresponding constant value for a single sphere is 3.) Multiplying the $\xi = 1$ value by 2 and squaring both, it is seen that the energy radiated along the x axis (the axis of the spheres) is negligible compared with that radiated along the y axis by a factor of 8000 approximately.

3. Heaving sphere near a wall

Here the centre of a sphere of radius a is at distance d (> a) from a vertical wall which lies in the plane $x = 0$. Defining bispherical co-ordinates (μ, η, θ) by

$$x = \frac{c\sinh\mu}{\cosh\mu-\cos\eta}, \quad (y,z) = \frac{c\sin\eta}{\cosh\mu-\cos\eta}(\sin\theta, \cos\theta), \tag{3.1}$$

the wall is given by $\mu = 0$, the sphere boundary by $\mu = \mu_0$, where

$$c = a\sinh\mu_0, \quad d = a\cosh\mu_0, \tag{3.2}$$

and the free surface by $\theta = \pm\frac{1}{2}\pi$ (except on the x axis, where $\eta = 0$ or π). The fluid region is $0 \leqslant \mu \leqslant \mu_0$, $0 \leqslant \eta \leqslant \pi$, $|\theta| \leqslant \frac{1}{2}\pi$ and the wall condition

$$\partial\phi/\partial\mu = 0 \quad \text{at} \quad \mu = 0 \tag{3.3}$$

requires that the velocity potential be an even function of μ. The situation is equivalent to that of equal spheres heaving with the same amplitude and phase. Conditions (1.3) and (1.6) on the limit potential $\phi_0(\mu, \eta, \theta)$ are now

$$\frac{\partial\phi_0}{\partial\mu} = -Uc\frac{\sinh\mu_0\sin\eta}{(\cosh\mu_0-\cos\eta)^2}\cos\theta \quad \text{at} \quad \mu = \mu_0, \tag{3.4}$$

$$\phi_0 = 0 \quad \text{when} \quad \sin\eta\cos\theta = 0.$$

The appropriate solution of Laplace's equation for ϕ_0 is of the form

$$\phi_0 = Uc(\cosh\mu-\cos\eta)^{\frac{1}{2}}\sum_{n=1}^{\infty} a_n\cosh\left(n+\tfrac{1}{2}\right)\mu P_n^1(\cos\eta)\cos\theta \tag{3.5}$$

(Morse & Feshbach 1953, p. 1299), the coefficients being determined by (3.4) and such that $a_n = o(\exp(-n\mu_0))$ as $n \to \infty$.

In virtue of the series expansion

$$\frac{\sin\eta}{(\cosh\mu-\cos\eta)^{\frac{3}{2}}} = 2\sqrt{2}\sum_{n=1}^{\infty}\exp\left[-\left(n+\tfrac{1}{2}\right)\mu\right]P_n^1(\cos\eta) \tag{3.6}$$

(Morse & Feshbach 1953, p. 1300) and the recurrence relation

$$(2n+1)\cos\eta P_n^1(\cos\eta) = nP_{n+1}^1(\cos\eta)+(n+1)P_{n-1}^1(\cos\eta) \quad (n \geqslant 1),$$

it is found that $\{a_n;\, n \geqslant 1\}$ satisfy the difference equation

$$(n+2)\, a_{n+1} \sinh\,(n+\tfrac{3}{2})\,\mu_0 - [(n+1)\sinh\,(n+\tfrac{3}{2})\,\mu_0 + n\sinh\,(n-\tfrac{1}{2})\,\mu_0]\, a_n$$
$$+ (n-1)\, a_{n-1} \sinh\,(n-\tfrac{1}{2})\,\mu_0 = 4\sqrt{2}\exp\,[-(n+\tfrac{1}{2})\,\mu_0]\sinh\mu_0 \qquad (n \geqslant 1). \qquad (3.7)$$

The apparent appearance of the undefined a_0 is nullified by the factor $n-1$, i.e. only one condition is required to determine a unique solution and this is that $a_n = o(\exp\,(-n\mu_0))$ as $n \to \infty$, in order that (3.5) be convergent throughout the fluid region. Unlike the corresponding torus problem (Davis 1975*b*), the sum of the coefficients on the left-hand side of (3.7) is non-zero; this is because P_n^1 has appeared instead of P_n. Since a closed-form solution is not available, (3.7) must be solved numerically. The convergence is improved by writing

$$a_n = -\frac{\sqrt{2}\exp\,[-(n+\frac{1}{2})\,\mu_0]}{\sinh\,(n+\frac{1}{2})\,\mu_0} + b_n \quad (n \geqslant 1), \qquad (3.8)$$

whence (3.7) becomes

$$(n+2)\, b_{n+1} \sinh\,(n+\tfrac{3}{2})\,\mu_0 - [(n+1)\sinh\,(n+\tfrac{3}{2})\,\mu_0 + n\sinh\,(n-\tfrac{1}{2})\,\mu_0]\, b_n$$
$$+ (n-1)\, b_{n-1} \sinh\,(n-\tfrac{1}{2})\,\mu_0 = -\frac{\sqrt{2}\exp\,[-(2n+1)\,\mu_0]}{\sinh\,(n+\frac{1}{2})\,\mu_0}\sinh\mu_0 \quad (n \geqslant 1). \qquad (3.9)$$

The virtual-mass coefficient is given, from (1.7), (3.1) and (3.4), by

$$V \sim \frac{3}{2\pi a^3 U^2}\int_0^{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (-\phi_0)_{\mu=\mu_0} \frac{Uc^2 \sinh\mu_0 \sin^2\eta}{(\cosh\mu_0 - \cos\eta)^3}\cos\theta\, d\theta\, d\eta$$
$$= -\tfrac{3}{4}\sinh^4\mu_0 \sum_{n=1}^{\infty} a_n \cosh\,(n+\tfrac{1}{2})\,\mu_0 \int_0^{\pi} P_n^1(\cos\eta)\frac{\sin^2\eta\, d\eta}{(\cosh\mu_0 - \cos\eta)^{\frac{5}{2}}}$$

after substituting (3.2) and (3.5). But, from (3.6),

$$\frac{\sinh\mu\sin\eta}{(\cosh\mu - \cos\eta)^{\frac{3}{2}}} = \frac{4\sqrt{2}}{3}\sum_{n=1}^{\infty}(n+\tfrac{1}{2})\exp\,[-(n+\tfrac{1}{2})\,\mu]\, P_n^1(\cos\eta),$$

whence $$\sinh\mu \int_0^{\pi}\frac{\sin^2\eta\, P_n^1(\cos\eta)}{(\cosh\mu - \cos\eta)^{\frac{5}{2}}}\,d\eta = \frac{4\sqrt{2}}{3}\, n(n+1)\exp\,[-(n+\tfrac{1}{2})\,\mu]. \qquad (3.10)$$

Then the expression for V simplifies to

$$V \sim -\sqrt{2}\sinh^3\mu_0 \sum_{n=1}^{\infty} n(n+1)\, a_n \exp\,[-(n+\tfrac{1}{2})\,\mu_0]\cosh\,(n+\tfrac{1}{2})\,\mu_0. \qquad (3.11)$$

Since the coefficients $\{a_n\}$ satisfy (3.7), this can be rewritten as

$$V \sim \sinh^3\mu_0 \sum_{n=1}^{\infty} n(n+1)\{8\exp\,[-(2n+1)\,\mu_0] + 3\sqrt{2}\, a_n \exp\,[-(n+\tfrac{1}{2})\,\mu_0]\sinh\,(n+\tfrac{1}{2})\,\mu_0\}$$
$$= \tfrac{1}{2} + 3\sqrt{2}\sinh^3\mu_0 \sum_{n=1}^{\infty} n(n+1)\, b_n \exp\,[-(n+\tfrac{1}{2})\,\mu_0]\sinh\,(n+\tfrac{1}{2})\,\mu_0$$

after substituting (3.8). Defining $\{B_n;\, n \geqslant 1\}$ such that $\sqrt{2}\, B_n = n(n+1)\, b_n \sinh^3\mu_0$, it follows that

$$V \sim \tfrac{1}{2} + 3\sum_{n=1}^{\infty} B_n\{1 - \exp\,[-(2n+1)\,\mu_0]\}. \qquad (3.12)$$

μ_1	d/a	V	V (Lamb)
1	1·5431	0·52620 (7)	0·52552
1·5	2·3524	0·50724 (4)	0·50720
2	3·7622	0·50176 (3)	0·50176
3	10·068	0·50009 (2)	0·50009
4	27·308	0·50000 (1)	0·50000

TABLE 1

The coefficients are determined by (3.9), which, by writing

$$\alpha_n = \sinh(n-\tfrac{1}{2})\mu_0/\sinh(n+\tfrac{3}{2})\mu_0,$$

can be simplified to

$$B_{n+1}-\left(\frac{n+1}{n}+\alpha_n\right)B_n+\frac{n+1}{n}\alpha_n B_{n-1} = -\frac{(n+1)\exp[-(2n+1)\mu_0]\sinh^4\mu_0}{\sinh(n+\frac{1}{2})\mu_0\sinh(n+\frac{3}{2})\mu_0} \quad (n\geqslant 1), \qquad (3.13)$$

where $B_0 = 0$ by definition and $B_n = o(n^2\exp(-n\mu_0))$ as $n\to\infty$. Then B_n is of the form $B_n = -B_n^{(1)}+B_1B_n^{(2)}$, where

$$B_{n+1}^{(1)}-\left(\frac{n+1}{n}+\alpha_n\right)B_n^{(1)}+\frac{n+1}{n}\alpha_n B_{n-1}^{(1)} = \frac{(n+1)\exp[-(2n+1)\mu_0]\sinh^4\mu_0}{\sinh(n+\frac{1}{2})\mu_0\sinh(n+\frac{3}{2})\mu_0} \quad (n\geqslant 1,\quad B_0^{(1)} = B_1^{(1)} = 0),$$

$$B_{n+1}^{(2)}-\left(\frac{n+1}{n}+\alpha_n\right)B_n^{(2)}+\frac{n+1}{n}\alpha_n B_{n-1}^{(2)} = 0 \quad (n\geqslant 1,\quad B_0^{(2)} = 0,\quad B_1^{(2)} = 1).$$

B_1 must be determined by applying the condition at infinity. Now

$$\alpha_n = \exp(-2\mu_0)+O(\exp(-2n\mu_0))$$

and the exact solution of

$$\lambda_{n+1}-\left(\frac{n+1}{n}+\exp(-2\mu_0)\right)\lambda_n+\frac{n+1}{n}\exp(-2\mu_0)\lambda_{n-1} = 0 \qquad (3.14)$$

is $\lambda_n = C\exp(-2n\mu_0)+D[n(\exp(2\mu_0)-1)-1]$, i.e. $\lambda_n-\lambda_{n-1}\to$ constant as $n\to\infty$. It is also seen why the substitution (3.8) improves the convergence, namely by making the right-hand side of the difference equation decay exponentially faster than both parts of the complementary function. The sequences $\{B_n^{(1)}\}$ and $\{B_n^{(2)}\}$ must have the same property and, defining

$$C^{(j)} = \lim_{n\to\infty}(B_n^{(j)}-B_{n-1}^{(j)}) \quad (j = 1,2),$$

it follows that $B_1 = C^{(1)}/C^{(2)}$. The number of iterations required to determine $C^{(j)}$ to a given accuracy is evidently a decreasing function of μ_0 and is indicated in brackets in table 1, obtained using a pocket calculator. Since the method depends on the exponential decay of $\exp(-n\mu_0)$, the number of iterations required increases rapidly as μ_0 decreases below 1. For example, twelve iterations proved to be insufficient for the case $\mu_0 = 0{\cdot}5$.

The last column of table 1 shows the value of V given by the approximate formula

$$V \simeq \tfrac{1}{2}(1+\tfrac{3}{16}a^3/d^3),$$

which is equivalent to that quoted by Lamb [1932, §99, equation (7)].

Recalling that $V \sim \frac{1}{2}$ when the wall is absent [equation (1.8)], it is seen that the sphere must be within a few radii of the wall for the latter to have a significant effect on the virtual-mass coefficient, which is evidently a decreasing function of μ_0 with limiting value as $\mu_0 \to 0$ given by (2.7).

4. Sphere within a hemispherical lake

Suppose that the radius of the lake is λa and that its centre is at a distance νa $(0 < \nu < \lambda - 1)$ from that of the heaving sphere of radius a. The case $\nu = 0$ has been studied rigorously (Davis 1975*a*) and appears here only as a limiting case. Since the fluid is now of finite extent, there is no radiation of wave energy and the level of the mean free surface oscillates with amplitude $U/\sigma(\lambda^2-1)$. Hence the linearization of condition (1.2) is consistent with that of (1.3).

Using the bispherical co-ordinates defined by (3.1), the lake boundary is at $\mu = \mu^*$ while $\mu = \mu_0$ on the sphere. From (3.2), the values μ_0 and μ^* are determined from the relations

$$\sinh\mu_0 = \lambda\sinh\mu^* = c/a, \quad \sinh(\mu_0-\mu^*) = \nu\sinh\mu^*.$$

Hence

$$\cosh\mu_0 = (\lambda^2-1-\nu^2)/2\nu, \quad \cosh\mu^* = (\lambda^2-1+\nu^2)/2\nu\lambda,$$

$$c^2/a^2 = (\lambda+\nu+1)(\lambda+\nu-1)(\lambda-\nu+1)(\lambda-\nu-1)/4\nu^2.$$

With the fluid now of finite extent, resonance is possible and, since the surface wave disturbance is essentially in a thin layer below F, the resonant frequencies are asymptotically those of the problem obtained by replacing the sphere and lake boundary by vertical circular cylinders which intersect F in the same circles. Removing the e^{-Kz} factor, these resonant frequencies are the eigenvalues of the problem

$$(\nabla^2+K^2)\psi = 0 \quad (\mu^* \leqslant \mu \leqslant \mu_0,\ -\pi < \eta \leqslant \pi),$$

$$\partial\psi/\partial\mu = 0 \quad \text{at} \quad \mu = \mu^*, \mu_0.$$

In the absence of resonance, the decomposition (1.5) of ϕ is possible, and the conditions on ϕ_0 are the same as in the previous section except that (3.3) is applied at $\mu = \mu^*$ instead of $\mu = 0$. Thus, proceeding as before, it readily follows that

$$\phi_0 = Uc(\cosh\mu-\cos\eta)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{[t_n\cosh(n+\frac{1}{2})(\mu-\mu^*)+u_n\cosh(n+\frac{1}{2})(\mu_0-\mu)]}{\sinh(n+\frac{1}{2})(\mu_0-\mu^*)} \times P_n^1(\cos\eta)\cos\theta. \quad (4.1)$$

The coefficients of $\cosh(n+\frac{1}{2})\mu$ and $\sinh(n+\frac{1}{2})\mu$ are chosen in this form for convenience. Equation (3.5) is recovered by setting $\mu^* = 0$ and $t_n = a_n\sinh(n+\frac{1}{2})\mu_0$, $u_n = 0$ $(n \geqslant 1)$.

In order that ϕ_0 has the given normal derivatives at $\mu = \mu^*, \mu_0$, the coefficients $\{t_n, u_n\}$ must satisfy the coupled second-order difference equations

$$(n+2)\,t_{n+1} - [(2n+1)\cosh\mu_0 + \coth(n+\tfrac{1}{2})\,\mu' \sinh\mu_0]\,t_n + (n-1)\,t_{n-1}$$
$$-\frac{\sinh\mu_0}{\sinh(n+\frac{1}{2})\,\mu'}\,u_n = 4\sqrt{2}\exp[-(n+\tfrac{1}{2})\,\mu_0 \sinh\mu_0 \quad (n \geqslant 1), \quad (4.2a)$$

$$(n+2)\,u_{n+1} - [(2n+1)\cosh\mu^* - \coth(n+\tfrac{1}{2})\,\mu' \sinh\mu^*]\,u_n + (n-1)\,u_{n-1}$$
$$+\frac{\sinh\mu^*}{\sinh(n+\frac{1}{2})\,\mu'}\,t_n = 0 \quad (n \geqslant 1), \quad (4.2b)$$

where $\mu' = \mu_0 - \mu^*$. These determine $\{t_n, u_n;\, n \geqslant 2\}$ in terms of t_1 and u_1, which must be chosen such that $t_n, u_n \to 0$ as $n \to \infty$. The convergence of the computation is improved by subtracting out the leading term $-\sqrt{2}\exp[-(n+\frac{1}{2})\,\mu_0]$ of t_n, as in the sphere/wall case.

The limit $\nu \to 0$, in which the sphere approaches the centre of the lake, is given by $\mu_0, \mu^* \to \infty$, $\mu' \to \log_e \lambda$. In this limit, a 'singular perturbation' situation occurs because the ratios of the coefficients of t_{n+1} and t_n in (4.2*a*) and of u_{n+1} and u_n in (4.2*b*) both tend to zero.

By comparison with (3.11), the virtual-mass coefficient is given by

$$V \sim -\sqrt{2}\sinh^3\mu_0 \sum_{n=1}^{\infty} n(n+1)\left[t_n \coth(n+\tfrac{1}{2})\,\mu' + \frac{u_n}{\sinh(n+\frac{1}{2})\,\mu'}\right]\exp[-(n+\tfrac{1}{2})\,\mu_0],$$

which by means of (4.2*a*) can be rewritten as

$$V \sim \tfrac{1}{2} + 3\sum_{n=1}^{\infty} T_n, \quad (4.3)$$

where $\quad T_n = n(n+1)\{\sqrt{2}\,t_n + 2\exp[-(n+\tfrac{1}{2})\,\mu_0]\}\exp[-(n+\tfrac{1}{2})\,\mu_0]\sinh^3\mu_0.$

Defining also

$$U_n = n(n+1)\sqrt{2}\,u_n \exp[-(n+\tfrac{1}{2})\,\mu^*]\sinh^3\mu_0,$$

the difference equations (4.2*a, b*) become

$$n(\coth\mu_0+1)\,T_{n+1} - [(2n+1)\coth\mu_0 + \coth(n+\tfrac{1}{2})\,\mu']\,T_n + (n+1)(\coth\mu_0 - 1)\,T_{n-1}$$
$$-[\coth(n+\tfrac{1}{2})\,\mu' - 1]\{U_n - 2n(n+1)\exp[-(2n+1)\,\mu_0]\sinh^3\mu_0\} = 0 \quad (n \geqslant 1), \quad (4.4a)$$

$$n(\coth\mu^*+1)\,U_{n+1} - [(2n+1)\,\mu^* - \coth(n+\tfrac{1}{2})\,\mu']\,U_n + (n+1)(\coth\mu^* - 1)\,U_{n-1}$$
$$+[\coth(n+\tfrac{1}{2})\,\mu' + 1]\{T_n - 2n(n+1)\exp[-(2n+1)\,\mu_0]\sinh^3\mu_0\} = 0 \quad (n \geqslant 1), \quad (4.4b)$$

where $T_0 = U_0$ by definition and T_n and U_n are exponentially small as $n \to \infty$.

In the limit $\mu_0 \to \infty$, $\mu' \to \log_e \lambda$, equations (4.4*a, b*) take the form

$$2nT_{n+1} - \left(2n+1+\frac{\lambda^{2n+1}+1}{\lambda^{2n+1}-1}\right)T_n - \frac{2U_n}{\lambda^{2n+1}-1} = \begin{cases} 0 & (n > 1), \\ -(\lambda^3-1)^{-1} & (n = 1), \end{cases}$$

$$2nU_{n+1} - \left(2n+1-\frac{\lambda^{2n+1}+1}{\lambda^{2n+1}-1}\right)U_n + \frac{2\lambda^{2n+1}T_n}{\lambda^{2n+1}-1} = \begin{cases} 0 & (n > 1), \\ \lambda^3(\lambda^3-1)^{-1} & (n = 1). \end{cases}$$

$\lambda = 4$		$\nu = 2$	
ν	V	λ	V
0	0.52381	4	0·54388 (20)
0·5	0·52388 (10)		
1.	0·52660 (10)	6	0·50903 (20)
1·5	0·53229 (20)		
2	0·54388 (20)	8	0·50338 (10)
2·5	0·56937 (50)		——

TABLE 2

The required solution is

$$T_1 = \frac{1}{2(\lambda^3-1)}, \quad U_1 = \frac{-\lambda^3}{2(\lambda^3-1)}; \quad T_n = U_n = 0 \quad (n > 1).$$

Then, from (4.3),

$$V \sim \tfrac{1}{2}[1+3/(\lambda^3-1)],$$

in agreement with the leading term of the rigorous result (1.9). Returning to consider finite values of μ_0, it is readily seen that the general solution of (4.4a,b) is such that, as $n \to \infty$, $T_n \sim \lambda_n$ and $U_n \sim \nu_n$, where the λ_n satisfy (3.14) and

$$n \exp(\mu^*)(\nu_{n+1}-\nu_n)-(n+1)\exp(-\mu^*)(\nu_n-\nu_{n-1}) = 2\lambda_n \sinh \mu^*.$$

The complementary sequence of this equation is similar to that of (3.14), namely

$$\nu_n = A\{n[1-\exp(-2\mu^*)]+1\}\exp(-2n\mu^*)+B.$$

But a particular solution is less easily found and the behaviour at large n of the full complementary sequence of (4.4a,b) is more complicated than that of the difference equations in the previous or following sections.

However, the virtual-mass coefficient can be computed to sufficient accuracy by truncating the infinite set of linear relations between terms of the sequences $\{T_n\}$ and $\{U_n\}$. This is equivalent to writing $T_n = U_n = 0$ for all $n > N$, where $2N$ is the number of equations retained. A single sequence $\{W_m;\ 1 \leqslant m \leqslant 2N\}$ is obtained by defining $W_{2n-1} = T_n$ and $W_{2n} = U_n$ $(1 \leqslant n \leqslant N)$. Then, from (4.4$a,b$), it is readily seen that the matrix of coefficients of $\{W_m\}$ is of band-diagonal form, all non-zero elements being confined to within two lines of the diagonal. Using the DGELB subroutine on an IBM 360 computer, the simultaneous equations were solved for various values of μ_0 and μ^* and the contributions of $\{T_n\}$ to V, given by (4.3), computed. Accuracy can be checked by increasing N; the value used is indicated in brackets in table 2, which displays values of V for various ν at given λ, showing the effect on V of moving the sphere from the centre towards the boundary of the lake, and for various λ at given ν, showing the effect of increasing the size of the lake.

5. Two separate spheres

Here the general case described in the introduction is considered. Using the bi-spherical co-ordinates defined by (3.1), the boundaries of the spheres S_1 and S_2 are given by $\mu = \mu_1$ and $\mu = -\mu_2$ respectively, where

$$a = c\,\mathrm{cosech}\,\mu_1, \quad b = c\,\mathrm{cosech}\,\mu_2. \tag{5.1}$$

If D is the distance between the centres of the spheres, then

$$D = c(\coth\mu_1 + \coth\mu_2),$$

and μ_1, μ_2 and c are determined, when a, b and D are given, by the formulae

$$\cosh\mu_1 = (D^2+a^2-b^2)/2aD, \quad \cosh\mu_2 = (D^2+b^2-a^2)/2bD,$$

$$c^2 = (D+a+b)(D+a-b)(D-a+b)(D-a-b)/4D^2.$$

The limit potential $\phi_0(\mu,\eta,\theta)$ is of the form

$$\phi_0 = Uc(\cosh\mu - \cos\eta)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{t_n \cosh(n+\frac{1}{2})(\mu+\mu_2) + u_n \cosh(n+\frac{1}{2})(\mu_1-\mu)}{\sinh(n+\frac{1}{2})(\mu_1+\mu_2)} \times P_n^1(\cos\eta)\cos\theta, \quad (5.2)$$

where the coefficients $\{t_n, u_n;\, n \geqslant 1\}$ are determined by (1.3) and (1.3a), which in the current co-ordinates become

$$\frac{\partial\phi_0}{\partial\mu} = -Uc\frac{\sinh\mu_1\sin\eta}{(\cosh\mu_1-\cos\eta)^2}\cos\theta \quad \text{at} \quad \mu = \mu_1, \quad (5.3)$$

$$\frac{\partial\phi_0}{\partial\mu} = \alpha Uc\frac{\sinh\mu_2\sin\eta}{(\cosh\mu_2-\cos\eta)^2}\cos\theta \quad \text{at} \quad \mu = -\mu_2. \quad (5.3a)$$

If $\mu_1 = \mu_2 = \mu_0$, ϕ_0 is the sum of $\frac{1}{2}(1+\alpha)$ times the solution (even in μ) found in §3 and $\frac{1}{2}(1-\alpha)$ times the corresponding odd solution, obtained by replacing (3.3) by the condition $\phi_0 = 0$ at $\mu = 0$.

When $\mu_1 \neq \mu_2$, the two sets of coefficients satisfy the coupled second-order difference equations

$$(n+2)t_{n+1} - [(2n+1)\cosh\mu_1 + \sinh\mu_1\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)]t_n + (n-1)t_{n-1} - \frac{u_n\sinh\mu_1}{\sinh(n+\frac{1}{2})(\mu_1+\mu_2)} = 4\sqrt{2}\exp[-(n+\tfrac{1}{2})\mu_1]\sinh\mu_1 \quad (n \geqslant 1),$$

$$(n+2)u_{n+1} - [(2n+1)\cosh\mu_2 + \sinh\mu_2\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)]u_n + (n-1)u_{n-1} - \frac{t_n\sinh\mu_2}{\sinh(n+\frac{1}{2})(\mu_1+\mu_2)} = 4\sqrt{2}\,\alpha\exp[-(n+\tfrac{1}{2})\mu_2]\sinh\mu_2 \quad (n \geqslant 1),$$

where t_1 and u_1 are to be chosen such that $t_n, u_n \to 0$ as $n \to \infty$. As in the earlier sections, it is helpful to write

$$\left.\begin{aligned} T_n &= n(n+1)\exp[-(n+\tfrac{1}{2})\mu_1]\{\sqrt{2}\,t_n + 2\exp[-(n+\tfrac{1}{2})\mu_1]\}, \\ U_n &= n(n+1)\exp[-(n+\tfrac{1}{2})\mu_2]\{\sqrt{2}\,u_n + 2\alpha\exp[-(n+\tfrac{1}{2})\mu_2]\}, \end{aligned}\right\} \quad (5.4)$$

whence the difference equations take the form

$$\begin{aligned} &n(\coth\mu_1+1)T_{n+1} - [(2n+1)\coth\mu_1 + \coth(n+\tfrac{1}{2})(\mu_1+\mu_2)]T_n \\ &\quad + (n+1)(\coth\mu_1-1)T_{n-1} - [\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)+1]\exp[-(2n+1)\mu_1]U_n \\ &\quad = -2n(n+1)[\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)-1]\{\exp[-(2n+1)\mu_1]+\alpha\}, \end{aligned} \quad (5.5a)$$

$$\begin{aligned} &n(\coth\mu_2+1)U_{n+1} - [(2n+1)\coth\mu_2 + \coth(n+\tfrac{1}{2})(\mu_1+\mu_2)]U_n) \\ &\quad + (n+1)(\coth\mu_2-1)U_{n-1} - [\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)+1]\exp[-(2n+1)\mu_2]T_n \\ &\quad = -2n(n+1)[\coth(n+\tfrac{1}{2})(\mu_1+\mu_2)-1]\{1+\alpha\exp[-(2n+1)\mu_2]\} \\ &\qquad (n \geqslant 1, T_0 = U_0 = 0). \end{aligned} \quad (5.5b)$$

The virtual-mass coefficients of the spheres, defined by (1.7), are given asymptotically by the formulae

$$\left.\begin{aligned} V_1 &\sim \tfrac{1}{2} + 3\sinh^3\mu_1 \operatorname{Re} \sum_{n=1}^{\infty} T_n, \\ V_2 &\sim \tfrac{1}{2} + 3\sinh^3\mu_2 \operatorname{Re} \sum_{n=1}^{\infty} U_n, \end{aligned}\right\} \tag{5.6}$$

which are obtained by substituting (5.1), (5.2), (5.3) and (5.3a) into (1.7), simplifying by means of (3.10) and the difference equations as in §§ 3 and 4, and finally substituting (5.4).

The linearity of (5.5) shows that T_n and U_n ($n \geqslant 1$) are of the form $T_n^* + \alpha T_n^{**}$ and $U_n^* + \alpha U_n^{**}$, where T_n^*, etc., are real functions of μ_1 and μ_2. The symmetry of (5.5) then implies that

$$T_n^{**}(\mu_1, \mu_2) = U_n^*(\mu_2, \mu_1), \quad U_n^{**}(\mu_1, \mu_2) = T_n^*(\mu_2, \mu_1).$$

Hence if $\{T_n^*, U_n^*\}$ are found by setting $\alpha = 0$ in (5.5), then $\{T_n^{**}, U_n^{**}\}$ can be obtained by interchanging μ_1 and μ_2 in the same calculation, a simpler adjustment to the computation than altering the inhomogeneous terms. It also follows from this symmetry that

$$V_2(\mu_1, \mu_2, \alpha) = V_1(\mu_2, \mu_1, \alpha^{-1})$$

as expected. Indeed, (5.6) have the form

$$\left.\begin{aligned} V_1 &= P(\mu_1, \mu_2) + (\operatorname{Re}\alpha)\, Q(\mu_1, \mu_2), \\ V_2 &= P(\mu_2, \mu_1) + (\operatorname{Re}\alpha^{-1})\, Q(\mu_2, \mu_1), \end{aligned}\right\} \tag{5.7}$$

where

$$\left.\begin{aligned} P(\mu_1, \mu_2) &= \tfrac{1}{2} + 3\sinh^3\mu_1 \sum_{n=1}^{\infty} T_n^*(\mu_1, \mu_2), \\ Q(\mu_2, \mu_1) &= 3\sinh^3\mu_2 \sum_{n=1}^{\infty} U_n^*(\mu_1, \mu_2). \end{aligned}\right\} \tag{5.8}$$

It remains to compute P and Q for various values of μ_1 and μ_2. At large n, the sequences $\{T_n\}$ and $\{U_n\}$ each satisfy equations like (3.14) (with μ_0 replaced by μ_1 and μ_2 respectively) and so in general are predominantly linear functions of n. T_1 and U_1 must be chosen to annihilate this linear behaviour, leaving only terms with exponential decay in $\{T_n, U_n\}$. Setting $\alpha = 0$ in all subsequent discussion of (5.5), the truncation procedure described in the previous section was first tried. However, when $\mu_1 \leqslant \mu_2$, the linear behaviour was not annihilated from the sequence $\{T_n^*\}$ and hence an extension of the method used to solve (3.13) was necessary.

The sequences can be decomposed in the form

$$T_n^* = T_n^{(1)} + T_1^* T_n^{(2)} + U_1^* T_n^{(3)}, \quad U_n^* = U_n^{(1)} + T_1^* U_n^{(2)} + U_1^* U_n^{(3)} \quad (n \geqslant 1),$$

where $\{T_n^{(1)}, U_n^{(1)}\}$ is the particular solution such that $T_1^{(1)} = 0 = U_1^{(1)}$ and the others are independent pairs of complementary sequences having

$$T_1^{(2)} = 1 = U_1^{(3)} \quad \text{and} \quad T_1^{(3)} = 0 = U_1^{(2)}.$$

After computing sufficient terms of each of the three pairs of sequences, the constants, identified as the initial terms T_1^* and U_1^*, are chosen to annihilate the linear terms in $\{T_n^*, U_n^*\}$, which sequences are then found and substituted into (5.8) to obtain the

μ_1	μ_2	a/b	D/b	$P(\mu_1, \mu_2)$	$Q(\mu_1, \mu_2)$	$P(\mu_2, \mu_1)$	$Q(\mu_2, \mu_1)$
0·25	1	4·6522	6·3414	0·50065	0·00296	0·50916	0·29840
0·5	1	2·2553	4·0862	0·50118	0·01103	0·50360	0·12653
1	1·5	1·8118	5·1482	0·50014	0·00550	0·50020	0·03270
1	2	3·0862	8·5244	0·50003	0·00121	0·50005	0·03559
1·5	2	1·7033	7·7691	0·50001	0·00160	0·50001	0·00790
0·5	0·5	1	2·2553	0·50671	0·06627		
1	1	1	3·0862	0·50067	0·02553		
1·5	1·5	1	4·7048	0·50004	0·00720		
2	2	1	7·5244	0·50000	0·00176		
		2	4	0·50095	0·01174	0·50218	0·09394
		2	6	0·50007	0·00347	0·50010	0·02778
		2	8	0·50001	0·00146	0·50001	0·01172
		3	6	0·50024	0·00347	0·50064	0·09380
		3	8	0·50004	0·00146	0·50007	0·03955
		4	6	0·50058	0·00348	0·50435	0·22302

Table 3

functions P and Q in (6.7). In deriving table 3 above, it was only in the first and last cases that it was necessary to consider more than 20 terms of each sequence.

According to (5.7), $P(\mu_1, \mu_2)$ measures the virtual-mass coefficient of S_1 when S_2 is fixed while $Q(\mu_1, \mu_2)$ gives the contribution due to the motion of S_2. The tabulated values indicate that the influence of a larger sphere on a smaller sphere is greater than vice versa, as expected on physical grounds.

The simple formulae given by Lamb [1932, §99, equation (6)] are equivalent to the approximations

$$P(\mu_1, \mu_2) \simeq \tfrac{1}{2}(1 + \tfrac{3}{4}a^3b^3/D^6) \simeq P(\mu_2, \mu_1),$$

$$Q(\mu_1, \mu_2) \simeq \tfrac{3}{4}b^3/D^3, \quad Q(\mu_2, \mu_1) \simeq \tfrac{3}{4}a^3/D^3.$$

Agreement is excellent for both values of Q but poor for P, particularly $P(\mu_2, \mu_1)$ when $\mu_2 > \mu_1$. The more accurate formulae of Basset [1888, §229, equation (43)] give good agreement for $P(\mu_1, \mu_2)$ but still cannot predict the above-listed values of $P(\mu_2, \mu_1)$. Evidently the method of successive images furnishes a poor approximation to the virtual mass of a smaller sphere heaving in the presence of a larger fixed sphere.

The work done in one cycle by S_1 on the fluid is, to first order,

$$\tfrac{4}{3}\pi^2 a^3 U^2 \rho (\operatorname{Im} \alpha)\, Q(\mu_1, \mu_2)$$

by comparison with (1.7), and, as explained in the introduction, must equal the first-order energy absorbed in one cycle by S_2, namely

$$-\tfrac{4}{3}\pi^2 b^3 |\alpha|^2 U^2 \rho (\operatorname{Im} \alpha^{-1})\, Q(\mu_2, \mu_1).$$

Consequently

$$a^3 Q(\mu_1, \mu_2) = b^3 Q(\mu_2, \mu_1),$$

i.e. $Q(\mu_2, \mu_1)/Q(\mu_1, \mu_2) = \sinh^3 \mu_2 / \sinh^3 \mu_1$ and $\sum_{n=1}^{\infty} U_n^*$ is symmetric in (μ_1, μ_2). So the effect of the motion of one sphere on the virtual-mass coefficient of the other is proportional to the cube of its radius, i.e. its volume.

Another physically expected result indicated by the calculations and proved to be true in the cylindrical case is the positivity of Q, which, owing to the time dependence being $e^{-i\sigma t}$, shows that the sphere with phase lag absorbs energy from the one oscillating in advance.

It is interesting to examine how V_1 and V_2, given by (5.7), depend on the phase of α. Suppose, for convenience, that $|\alpha| = 1$. Since $Q > 0$, the maximum values of V_1 and V_2 both occur at $\alpha = 1$, while the minimum values are both at $\alpha = -1$. In both these cases, there is no exchange of energy. The values of $P+Q$ given for $\mu_1 = \mu_2$ in table 3 agree with the corresponding values of V in table 1.

If the phase difference is a quarter of a period (α pure imaginary), then the motion of one sphere has no effect on the virtual-mass coefficient of the other. However, in this case, the exchange of energy per cycle has its maximum (for given $|\alpha|$).

The forces required to maintain the forced heaving motions have been shown to be maximum when $\alpha = 1$ and minimum when $\alpha = -1$. However the pressures due to the fluid motion are such that these in-phase and out-of-phase oscillations are respectively stable and unstable in the following sense. If the phase difference changes slightly from 0 or π, then energy exchange takes place and, since it favours the lagging sphere, tends in the respective cases to decrease or increase the deviation of the phase of α.

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